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Canonical icosahedral quasilattices for the F-phase generated by coherent phases in physical space

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Abstract. We examine the two canonical quasicrystal tilings for the icosahedral F-phase from the point of view of Meyer's theory of ε-duals. Using this we develop all the steps of an algorithm for creating the vertex sets of these tilings by outward growth from a starting seed.

1. Introduction

In the projection method for quasilattices, points projected into physical (parallel) space are accepted if their projections into window (perpendicular) space fall into a bounded region. The coherent phase method proposed in [2] uses the almost periodicity of the quasilattice and selected continuous characters on the physical space to dispose of all reference to window space and to generate the points of the quasilattice by a systematic growth algorithm based entirely in physical space.

In this paper we explore the coherent phase method in the context of the icosahedral F-phase and the two canonical tilings associated with this phase. We introduce all the concepts of the coherent phases method and then, utilizing the known windows of the canonical tilings, determine a finite set of continuous characters χ_μ on $\mathbb{R}^3$, indexed by suitably selected points μ from reciprocal space, and a finite set of controlling parameters $\varepsilon > 0$, which completely specify the method for these two classes of tilings. As a result we obtain a simple algorithm that generates the vertex sets of these tilings.

The method relies on two separate concepts. The first is Meyer's theory of duality [1] which associates to each quasilattice Λ appearing from the cut and project technique and each $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon < 2$, an ε-dual quasilattice Λ^ε consisting of continuous characters for which Λ is an approximate set of periods. The second, which is almost self-evident for convex tilings, is the possibility of creating the quasilattice of vertices by outward growth along sequences of edges of the tiling starting from some fixed starting vertex. We call this amenability.

In section 2 we recall the definitions involved with ε-duality and in section 3 we introduce the concepts and terminology that we need for amenability. In section 4 we illustrate how duality and amenability are combined into the coherent phases method by illustrating it on the simple one-dimensional Fibonacci quasilattice. This serves as a prelude to the analysis of the method as applied to the tilings of the type $\mathcal{T}^{*(2F)}$ (in section 5) and $\mathcal{T}^{(2F)}$ (in section 6).

We have implemented these algorithms in Mathematica. Several examples of the graphical output of these programs are provided as illustrations.

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2. Meyer sets

Let us define the cut and project method for the generation of a quasilattice. Let $L \subset \mathbb{R}^{m+k}$ be a lattice with symmetry G and let $p_\parallel$, $p_\perp$ be parallel, orthogonal projections respectively

$$p_\parallel : \mathbb{R}^{m+k} \longrightarrow \mathbb{R}^m \qquad p_\perp : \mathbb{R}^{m+k} \longrightarrow \mathbb{R}^k \tag{1}$$

satisfying

$$\mathrm{Ker}(p_\parallel) \cap L = (0) \text{ and } p_\perp(L) \text{ is dense in } \mathbb{R}^k \tag{2}$$

and

$$p_\parallel \text{ and } p_\perp \text{ are } G\text{-invariant.} \tag{3}$$

$\mathbb{R}^m$ and $\mathbb{R}^k$ are referred to as physical and orthogonal space respectively. Let P be a bounded region of $\mathbb{R}^k$ that contains a non-empty open subset of $\mathbb{R}^k$. Let $\gamma \in \mathbb{R}^k$. Then we define the quasilattice (model set) $\Lambda(P, \gamma)$ by

$$\Lambda = \Lambda(P, \gamma) := \{p_\parallel(x) | x \in L,\, p_\perp(x) \in P + \gamma\}. \tag{4}$$

Hereafter we use the notation: $p_\parallel(x) \equiv x$; $p_\perp(x) \equiv x^*$. The subsets of $\mathbb{R}^m$ generated by the cut and project method are examples of Meyer sets [1, 2] to be defined below.

Meyer [1] introduced the concept of harmonious sets in the context of locally compact Abelian groups. Moody and Patera [2] (see also [3, 4]) used a slightly stronger concept and restricted themselves to the case of Euclidean space $\mathbb{R}^m$, coining the term Meyer set. In this sense, we present here two of the many equivalent definitions of Meyer sets. The first shows that Meyer sets are generalizations of lattices. The second is the one that is relevant for our purposes here.

2.1. First definition of a Meyer set

A subset $\Lambda \subset \mathbb{R}^m$ is a Meyer set iff:

(i) it is Delaunay;

(ii) there is a finite set F such that $\Lambda - \Lambda \subset \Lambda - F$.

The second definition involves the duality (reciprocity) theory. Suppose $\Lambda \subset \mathbb{R}^m$ is any subset. We let $[\Lambda]$ denote the subgroup of $\mathbb{R}^m$ generated by Λ. A character, or *algebraic character* is a homomorphism

$$\chi : [\Lambda] \longrightarrow U(1) := \{z \in \mathbb{C} | |z| = 1\}. \tag{5}$$

We also call χ a character on Λ (even though Λ is not itself a group in general). The set of continuous characters $\mathbb{R}^m \longrightarrow U(1)$ are all of the form

$$\chi_\mu : x \longmapsto \mathrm{e}^{2\pi \mathrm{i} \mu \cdot x} \tag{6}$$

for some $\mu \in \mathbb{R}^m$. The set of all these characters forms the dual group $\widehat{\mathbb{R}^m}$. The mapping $\mu \longrightarrow \chi_\mu$ allows us to identify $\mathbb{R}^m$ and $\widehat{\mathbb{R}^m}$ if we wish.

Let $\Lambda \subset \mathbb{R}^m$ be a Delaunay set. Let χ be an arbitrary algebraic character on Λ and let $\varepsilon > 0$ be arbitrary. $\chi_\mu \in \widehat{\mathbb{R}^m}$ is an *ε-uniform approximation of χ on Λ* if,

$$\text{for all } x \in \Lambda \qquad |\chi_\mu(x) - \chi(x)| < \varepsilon \qquad \varepsilon < 2. \tag{7}$$

2.2. Second definition of a Meyer set

A Delaunay set Λ is a Meyer set if and only if for each character $\chi : [\Lambda] \longrightarrow U(1)$ and for each $\varepsilon > 0$ there exists an ε-uniform approximation $\chi_\mu \in \widehat{\mathbb{R}^m}$ of χ on Λ (μ depends on Λ, χ, and ε).

Let $\chi(x) = 1$ be the trivial character. For $\varepsilon > 0$ define the *ε-dual* of Λ by

$$\Lambda^\varepsilon := \left\{ \chi_\mu \in \widehat{\mathbb{R}^m} | \chi_\mu \begin{array}{l} \text{is a } \varepsilon\text{-uniform approximation} \\ \text{of the trivial character on } [\Lambda] \end{array} \right\}. \tag{8}$$

Thus $\chi_\mu \in \Lambda^\varepsilon \Leftrightarrow |e^{2\pi i \mu x} - 1| < \varepsilon$ for all $x \in \Lambda$. If Λ is a Meyer set and $\varepsilon < 2$, then Λ^ε is itself a Meyer set in $\widehat{\mathbb{R}^m} \simeq \mathbb{R}^m$ by the natural identification of χ_μ with μ. We will illustrate this fact in an example of the Fibonacci quasilattice in section 4.

In [2], based on the second definition, a method was developed for generating quasicrystals. We call it the *coherent phases* method. The idea in its simplest formulation is that although in principle we need to know all of Λ^ϵ to determine Λ, in the context of additional information we may be able to use only finitely many elements of Λ^ϵ and inequalities such as (7) to correctly decide which of the various points do or do not belong to Λ.

3. The coherent phases method

If Λ is a lattice, then its dual (reciprocal) lattice Λ^0 can be completely determined by a finite number of elements of Λ^0, namely a basis of Λ^0. If Λ is a quasilattice produced as the point set arising from the cut and project method, then, under mild assumptions it can be completely determined by a finite number of elements of Λ^ε, approximants of the algebraic character χ on Λ (for some $\epsilon > 0$).

The coherent phases method, in cases of the quasilattices related to the *canonical* tilings [7], obtained by the projection from the root lattices, transforms the information of the window and all its substructure into a finite set of continuous characters, an ideal local configuration, and a starting seed (see below): it eliminates the orthogonal (window)-space and provides a simple procedure which generates the points of the quasilattice by inspection of their position in parallel space. The points are obtained in an organized way outward from the starting seed.

We now define the terms *amenability*, *starting seed*, and *ideal local configuration*.

Let $\Lambda \subset \mathbb{R}^m$ be a quasilattice and let $S \subset \mathbb{R}^m$ be a finite subset. We define an equivalence relation $\sim$ on Λ by setting $x \sim y$ if $x - y \in \pm S$ and then taking the transitive closure of this relation, i.e. $x \sim y$ if and only if there exist $x_0 = x, x_1, \ldots, x_{n-1}, x_n = y$, all elements of Λ, so that $x_i - x_{i+1} \in \pm S$ for all $i = 0, 1, \ldots, n-1$. Λ is *weakly amenable* to S if there are only finitely many equivalence classes in Λ relative to $\sim$.

Amenability can be seen as another way in which to generalize the finite generation of a lattice. Suppose that Λ is weakly amenable to S, and suppose that $X \subset \Lambda$ is a finite set so that each $x \in \Lambda$ is equivalent (under $\sim$) to some element of X. Then given any point $u \in \Lambda$ we can find a path in Λ from some point $x \in X$ to u where each step of the path is obtained by adding or subtracting a vector of S.

We say that Λ is *(strongly) amenable* to S if X can be chosen so that for each $z \in \Lambda$ we can find a path $x_0 = x, x_1, \ldots, x_n = z$ where:

(i) $x_{i+1} - x_i \in S$, $i = 0, 1, \ldots, n-1$;

(ii) $|x_{i+1} - x_0| > |x_i - x_0|$, $i = 0, 1, \ldots, n-1$.

Effectively the path uses only vectors of S and is outwardly expanding from X. The set X is called the *starting seed*, and the set S is called the *ideal local configuration*. For each $z \in \Lambda$ we define $S(z) := \{s \in S | z + s \in \Lambda\}$. This is the *local configuration* at z.

Good examples of this occur in the case of quasiperiodic tilings $\mathcal{T}$ of $\mathbb{R}^m$ by convex tiles. Then the set Λ of vertices of the tiling is amenable to the set S of all edge vectors. The finiteness of the set X is guaranteed by the finite number of local configurations.

The concept of amenability was introduced in [2, 3] and a number of conditions assuring it were established. In the context of this paper, where we are given the *a priori* existence of the tilings, amenability is obvious from the remarks we have just made (and, in fact, would be very hard to establish in any other way). In the case of $\mathcal{T}^{*(2F)}$ the edge vectors fall into a single set of 60 vectors (see section 4). In the case of $\mathcal{T}^{(2F)}$ the set of edge vectors decomposes into three classes, $S = S_a \cup S_b \cup S_c$, as explained in section 5.

The elements of the method are as follows.

(1) The *ideal local configuration* is a set of vectors S which serve as the basic generators for the growth: $x_i + s = x_{i+1}$, $s \in S$.

(2) A *starting seed* X of Λ from which further growth will proceed.

(3) The generation or *growth process*: if z is an existing (already created) point of Λ, then the set of points $x = z + s$, $s \in S$ such that $|z + s - x_0| > |z - x_0|$, (in the *outward* direction) are new *potential* points of Λ.

(4) The *selection process* is a decision process (based on a finite set of $\{\chi_\mu, \varepsilon\}$ by inequalities of the type (7)) which selects from the potential points those that will be accepted as points of Λ. The rules of the selection process are also called *coherent phase conditions* [2–4].

We use the coherent phases method in order to generate the quasilattices corresponding to the vertices of the tilings $\mathcal{T}^{*(2F)}$ [5] and $\mathcal{T}^{(2F)}$ [6].

4. Generation of the Fibonacci quasilattice by the coherent phases method

In order to illustrate in more detail how the method works, we apply it to the simplest example, the Fibonacci quasilattice. The Fibonacci quasilattice is defined by the projection from $\mathbb{Z}^2$ with the basis $\{e_i, i = 1, 2 | (e_i \cdot e_j) = \delta_{ij}\}$ to $\mathbb{R}^1$ embedded in $\mathbb{Z}^2$ such that $\tan\phi = 1/\tau$ where ϕ is the angle between $\mathbb{R}^1$ and x-axis of $\mathbb{Z}^2$, and the window P in $\mathbb{E}_\perp$ is an interval of length $\tau^2 K$, where $K = 1/\sqrt{\tau + 2}$ and $\tau = (1 + \sqrt{5})/2$, i.e. Λ $(P = \tau^2 K)$.

(1) The ideal local configuration consists of two short-edge vectors $\pm K$ and two long-edge vectors $\pm\tau K$.

(2) The starting seed is one point $\{0\}$. Let $\gamma = 0$.

(4) The algebraic character is chosen to be $\chi(x) = 1$. The continuous character $\chi_\mu = \mathrm{e}^{2\pi \mathrm{i}\mu\cdot x}$ and the inequality of type (7)

$$x \in \Lambda \qquad \text{if } |\mathrm{e}^{2\pi \mathrm{i}\mu\cdot x} - 1| < \varepsilon \tag{9}$$

are defined when μ is determined. The equation can be rewritten in orthogonal space

$$|\mathrm{e}^{-2\pi \mathrm{i}\mu^*\cdot x^*} - 1| < \varepsilon \tag{10}$$

since $\mathrm{e}^{2\pi \mathrm{i}(\mu\cdot x + \mu^*\cdot x^*)} = 1$†. The left-hand side of inequality (10) defines a function $f_{\mu^*}(x^*)$ that is periodic in the variable x^* and the inequality of (10) decomposes orthogonal space into a set of periodically repeated parallel intervals of which the one containing 0 is to be our window P. The growth process by short- and long-edge vectors appears in orthogonal space as steps along (dual) short and long vectors respectively. In order for none of these

† Note that the normalization of projections (1) and consequently the definition of * are slightly different than in [2–4].

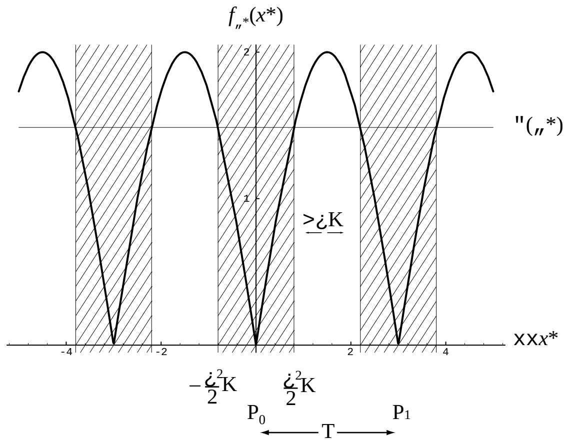

Figure 1. The non-overlap condition.

steps, in orthogonal space, to jump from window P into one of the parallel repeated copies of itself, the period T of the function $f_{\mu^*}(x^*)$ should satisfy the *non-overlap condition*

$$T > \tau^2 K/2 + \tau K + \tau^2 K/2 = \tau^3 K \tag{11}$$

(see figure 1). The restriction with respect to μ^* is

$$|\mu^*| < (3\tau - 4)K \tag{12}$$

and represents the window condition in the dual, reciprocal space. The length of the window P^ε is $2K(3\tau - 4)$. It is clear that Λ^ε is itself a Meyer set. For any fixed value of allowed μ^*, $\mu^* \in P^\varepsilon$,

$$|\mathrm{e}^{-2\pi \mathrm{i}\mu^* \cdot x^*} - 1| = 2|\sin \pi x^* \cdot \mu^*| < \varepsilon(\mu^*). \tag{13}$$

The value of $\varepsilon(\mu^*)$ can be determined by setting for x^* its maximal length in the window P:

$$\varepsilon(\mu^*) = 2\left|\sin \pi \frac{\tau^2}{2} K \cdot mu^*\right|. \tag{14}$$

Hence, the inequality of the type (7) for this case becomes

$$x \in \Lambda \qquad \text{if } |\mathrm{e}^{2\pi \mathrm{i}\mu \cdot x} - 1| < 2\left|\sin \pi \frac{\tau^2}{2} K \cdot \mu^*\right| \tag{15}$$

for $\mu \in \Lambda^\varepsilon$, i.e. $\mu^* \in P^\varepsilon$.

(3) In inequality (15), if z is an existing (already created) point of Λ, then the set of points

$$x = z + s \qquad s \in S = \{\pm K, \pm \tau K\} \tag{16}$$

such that $|z + s - x_0| > |z - x_0|$ are new potential points of Λ to be accepted. The acceptance condition is

$$z + s \in \Lambda \qquad \text{if } |\mathrm{e}^{2\pi \mathrm{i}\mu \cdot (z+s)} - 1| < 2\left|\sin \pi \frac{\tau^2}{2} K \cdot \mu^*\right|. \tag{17}$$

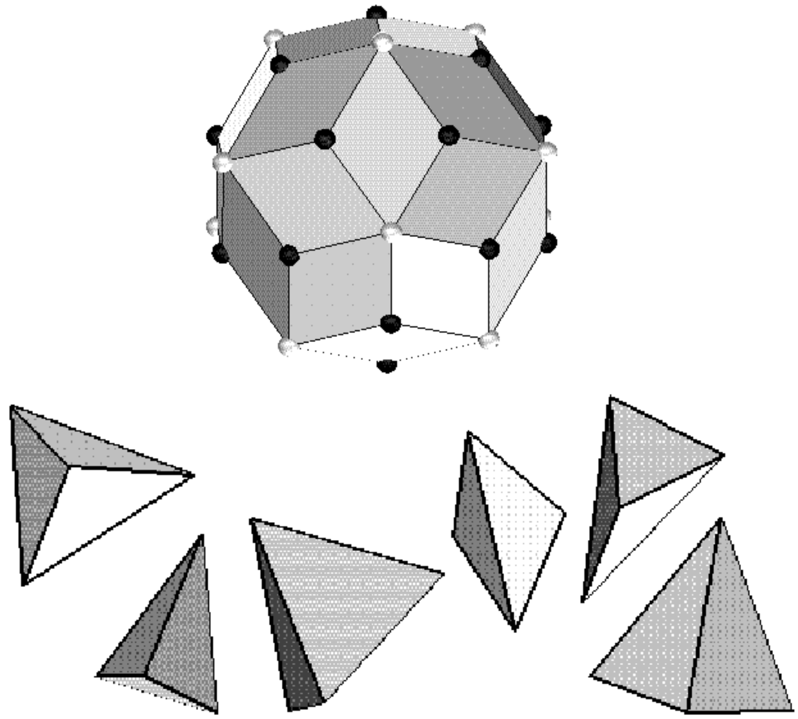

Figure 2. The window for the tiling $\mathcal{T}^{*(2F)}$ and the tiles.

5. Generation of the quasilattice related to the tiling $\mathcal{T}^{*(2F)}$

The window P for the tiling $\mathcal{T}^{*(2F)}$ [5, 7] and the related quasilattice is the Voronoi cell of the D_6 lattice icosahedrally projected to $\mathbb{E}_\perp$. With the scale, such that the basis for $\mathbb{Z}^6$ is $\{e_i, i = 1, \ldots, 6 | (e_i \cdot e_j) = \delta_{ij}\}$, the window is a triacontahedron with the edges along the five-fold direction of edge length ⑤ $= 1/\sqrt{2}$. We denote it by $T^{⑤}$. The tiles of the tiling $\mathcal{T}^{*(2F)}$ are six tetrahedra with all edges along the two-fold directions with two lengths, ② and τ②, ② $= \sqrt{2}K$. Their vertices are icosahedrally projected D_6 lattice points, see figure 2. There is no globally icosahedrally symmetric tiling in the class of the tilings $\mathcal{T}^{*(2F)}$.

In case of the quasilattice related to $\mathcal{T}^{*(2F)}$ we use the following elements.

(1) The chosen *ideal local configuration*

$$S = \{30 \text{ edge vectors ② and } 30 \text{ edge vectors } \tau②\}. \tag{18}$$

(2) For the *starting seed* of Λ we only need one point that we put to be $x_0 = 0$. The γ parameter from P is chosen so that $S(x_0)$ consists of the 30 edge vectors τ②, i.e. the γ parameter is taken from the coding polytope in $P = T^{⑤}$ related to this particular vertex configuration of the tiling $\mathcal{T}^{*(2F)}$. For the vertex configurations see [8].

(4) The algebraic character for the *selection process* is chosen as

$$\chi(x) = 1 \qquad \text{for all } x \in \Lambda. \tag{19}$$

The finite set of Λ^ε is chosen as follows.

We denote the icosahedrally projected D_6 lattice by F (or $2F$)-module [7] or M_3; the icosahedrally projected dual (reciprocal) lattice D_6^ω by I-module [7] or M_3^0

$$M_3 = \sum_{\text{roots}} \mathbb{Z}\alpha \qquad M_3^0 := \left\{ \mu \in \sum \mathbb{Q}[\tau]\alpha \,|\, \mu \cdot x + \mu^* \cdot x^* \in \mathbb{Z} \forall x \in M_3 \right\} \tag{20}$$

where μ and x are in parallel space $\mathbb{E}_\parallel$, μ^* and x^* are in orthogonal space $\mathbb{E}_\perp$. From the non-overlap condition we get the following restriction on the absolute value for allowed μ^*'s in orthogonal space in the two-fold directions:

$$|\mu^*| < \frac{\tau + 2}{2\tau^3} ②. \tag{21}$$

These restrictions define the window $T^{\frac{\tau+2}{\tau^5}⑤}$ in μ^*–(orthogonal) space, i.e. a quasilattice, or Meyer set in dual (reciprocal) space, in μ–(parallel) space (see figure 3). From the corresponding quasilattice we choose one μ for each of the 15 (30) two-fold directions, such that μ^* is small enough. The choice of the μ's is not unique. However, once chosen they define both the value of ε and the ε–uniform characters χ_μ. Our chosen representative μ in $\mathbb{E}_\parallel$ is

$$\mu = (2\tau + 1, 0, 0)② \tag{22}$$

and in $\mathbb{E}_\perp$

$$\mu^* = (\tau - 2, 0, 0)②. \tag{23}$$

For the chosen μ^* and the window condition we determine the ε of inequality (7). The window $T^{⑤}$ leads to the value of ε

$$\varepsilon = 2 \sin \frac{\pi}{\tau + 2} \approx 1.527 \tag{24}$$

for all two-fold directions. The final inequalities, to be checked for all 15 μ's (μ^*'s), are

$$|\mathrm{e}^{-2\pi \mathrm{i} \mu \cdot x} \mathrm{e}^{-2\pi \mathrm{i} \mu^* \cdot \gamma} - 1| < 2 \sin \frac{\pi}{\tau + 2} \tag{25}$$

where $\gamma = \frac{1}{14}(3\tau - 3, \tau - 2, -2\tau + 3)②$. The choice of γ ensures that the starting configuration is as described in step (2) (see figure 4).

(3) The generation or *growth process*: if z is an existing (already created) point of Λ then the set of points

$$x = z + s \qquad s \in S \tag{26}$$

are new *potential* points of Λ. Inequality (25) becomes

$$|\mathrm{e}^{-2\pi \mathrm{i} \mu \cdot (z+s)} \mathrm{e}^{-2\pi \mathrm{i} \mu^* \cdot \gamma} - 1| < 2 \sin \frac{\pi}{\tau + 2}. \tag{27}$$

Potential points that satisfy all 15 distinct conditions of inequalities (27) are accepted. We generate the quasilattice of the tiling $\mathcal{T}^{*(2F)}$ (see figure 5).

6. Generation of the quasilattice related to the tiling $\mathcal{T}^{(2F)}$

The Delaunay cells of the D_6 lattice projected to $\mathbb{E}_\perp$ are the acceptance domains for the quasilattice $\mathcal{T}^{(2F)}$ [6, 7]. A dodecahedron of edge length ② and two icosahedra with edges ② and τ② are acceptance domains for translationally inequivalent classes of holes of the lattice D_6. Let us denote representatives in these three classes of holes in six-dimensional space by $a = \frac{1}{2}(1\,1\,1\,1\,1\,1)$, $b = (1\,0\,0\,0\,0\,0)$ and $c = \frac{1}{2}(\bar{1}\,1\,1\,1\,1\,1)$, respectively. In $\mathbb{E}_\parallel$, there are three globally icosahedrally symmetric tilings seen from the vertices of the type

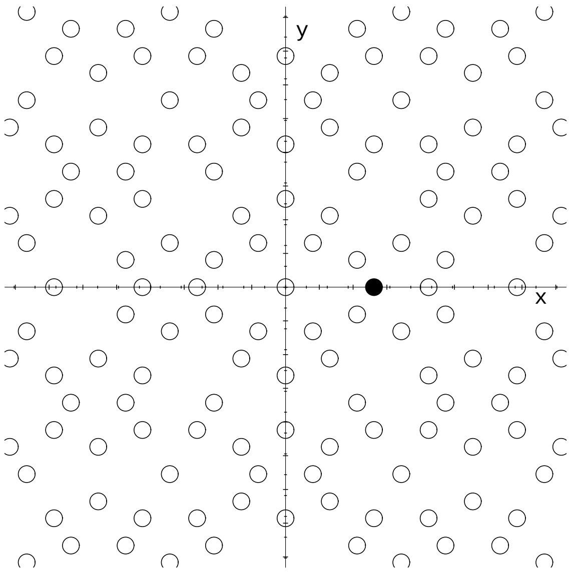

Figure 3. Meyer set or quasilattice $\Lambda^{\varepsilon}(T^{\frac{\tau+2}{\tau^5}\circled{5}})$ in reciprocal space, μ-space. The plane of the figure is orthogonal to a two-fold direction and passes through $\mu = 0$. The representative μ along a two-fold direction is marked.

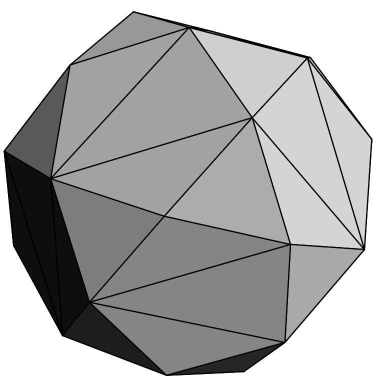

Figure 4. Vertex configuration of the tiling $\mathcal{T}^{*(2F)}$ corresponding to the chosen starting seed and γ parameter.

a, b and c, respectively. It is these three tilings that we will study here. The tiles of the tilings are obtuse and acute rhombohedra of the same shape (edge length ⑤) as those of the primitive tiling, $\mathcal{T}^{(P)}$ [9], but decorated by the vertices a (black circle) and c (white circle), as in figure 6, and four pyramids, each with base congruent to the rhombus face of the rhombohedra. The pyramid tops are of type b (grey circle) and their side–edges are either along the five-fold or three-fold directions. The standard length along the three-fold direction is ③ $= \sqrt{\frac{3}{2}(\tau + 2)}$.

In case of the quasilattice related to $\mathcal{T}^{(2F)}$ we use the following elements.

(1) The *ideal local configuration* consists of all edge vectors of the tiling $\mathcal{T}^{(2F)}$. It is given as follows. Each type of hole $z = a$, b and c gets its own ideal local configuration,

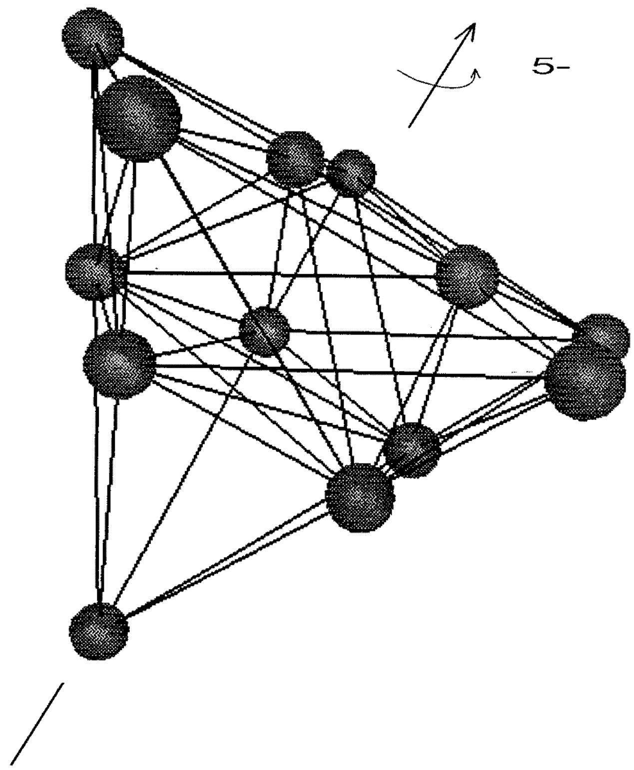

Figure 5. Points and edges produced by the coherent phases method in two steps for the quasilattice related to the tiling $\mathcal{T}^{*(2F)}$. A five-fold direction (not a symmetry) is marked.

i.e. the points x to be tested by an inequality of the type (7) are $x = z + s_z$, where $s_z \in S_z$

$$S_a = \left\{ \begin{array}{l} \bullet\text{———}\textcircled{5}\text{———}\circ \\ \bullet\text{—}\tfrac{1}{\tau}\textcircled{5}\text{—}\odot \\ \bullet\text{————}\tau\textcircled{3}\text{————}\odot \end{array} \right\} \tag{28}$$

$$S_b = \left\{ \begin{array}{l} \odot\text{—}\tfrac{1}{\tau}\textcircled{5}\text{—}\bullet \\ \odot\text{——————}\tau\textcircled{5}\text{——————}\circ \\ \odot\text{———}\textcircled{3}\text{———}\circ \\ \odot\text{————}\tau\textcircled{3}\text{————}\bullet \end{array} \right\} \tag{29}$$

$$S_c = \left\{ \begin{array}{l} \circ\text{———}\textcircled{5}\text{———}\bullet \\ \circ\text{——————}\tau\textcircled{5}\text{——————}\odot \\ \circ\text{———}\textcircled{3}\text{———}\odot \end{array} \right\} \tag{30}$$

where for example $\bullet\text{———}\textcircled{5}\text{———}\circ$ stands for the 12 edge vectors along five-fold directions of length $\textcircled{5}$, which lead from a hole of type a ($z = a$) to a hole of type c ($z + s_a = c$). Along three-fold directions we always have 20 edge vectors of the lengths $\textcircled{3}$ or $\tau\textcircled{3}$. Holes of type a are black, of type b are dotted circles, and of type c are white. In the figures the holes of type b are grey†.

† In the electronic version of the article, in figures 8, 9 and 10 the holes of type a are black, type b are violet and type c are yellow.

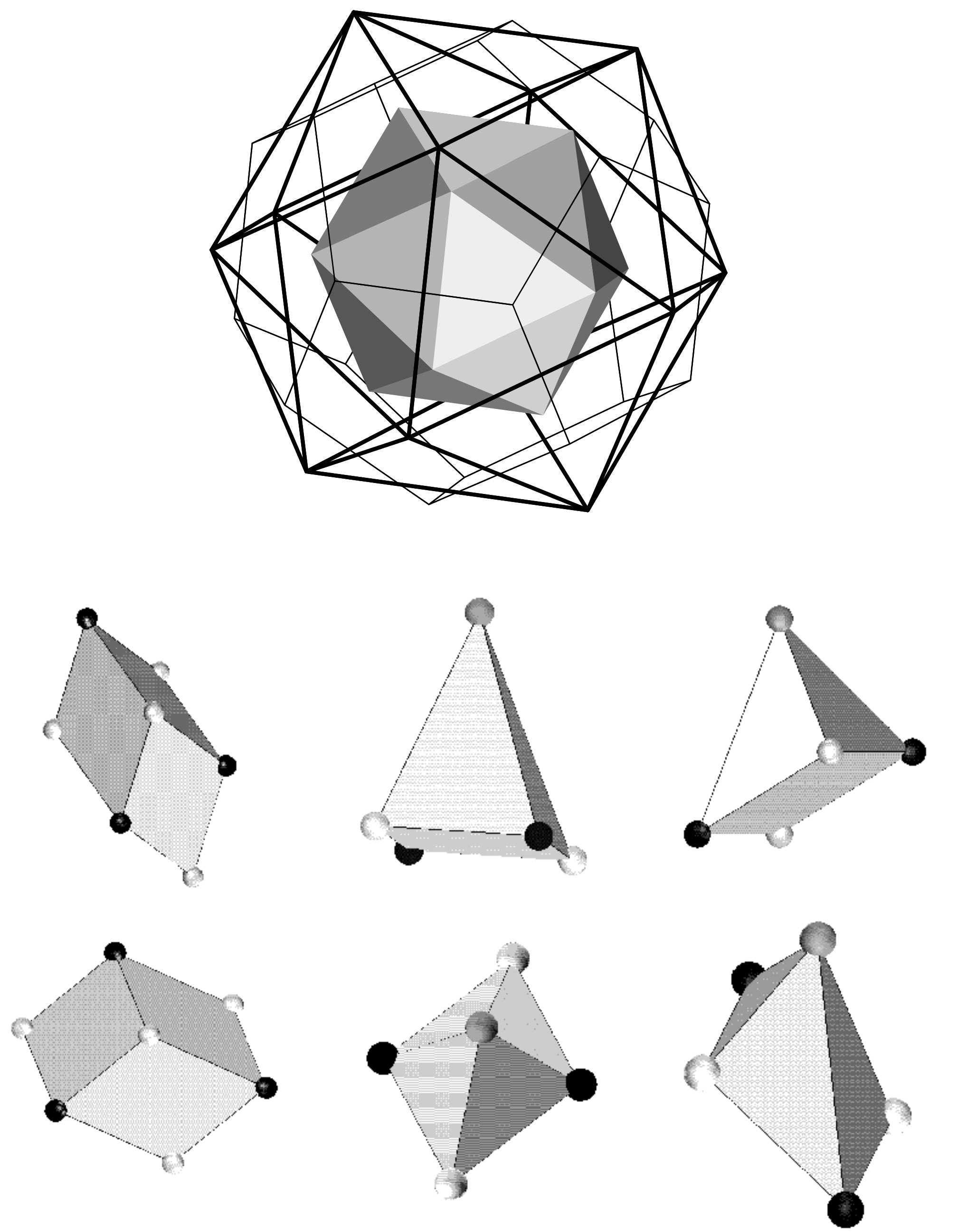

Figure 6. The windows for the tiling $\mathcal{T}^{(2F)}$ and the tiles.

(2) The *starting seeds* for the holes in Λ of types a, b and c are given in the figure captions for the examples of the constructed quasilattices (see figures 8–10).

(4) The choice of the *algebraic character* for the *selection process* by inequalities of

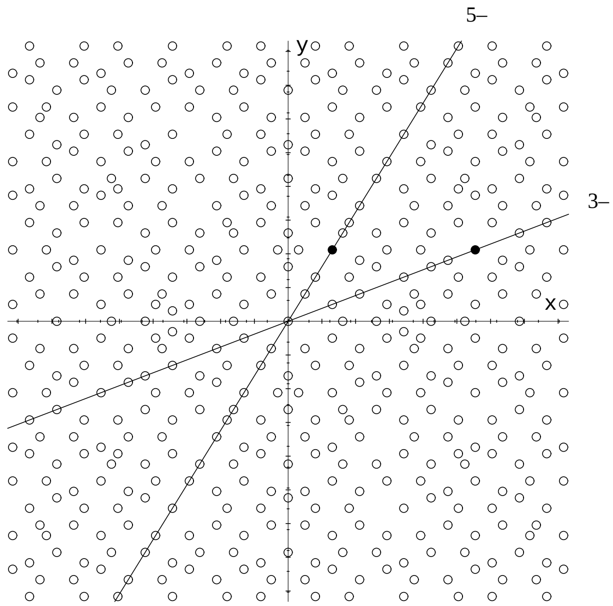

Figure 7. Meyer set or quasilattice Λ^{ε} (ball with radius $R = 0.373$) in reciprocal space, μ-space. The plane of the figure is orthogonal to a two-fold direction and passes through $\mu = 0$. One representative μ along a three-fold and another along a five-fold directon are marked.

the type (7) depends on which type of hole ($x = a$, b, or c) is to be tested for acceptance and on the type of the starting hole ($x_0 = a_0$, b_0, or c_0)

$$\chi(x) = \mathrm{e}^{2\pi \mathrm{i}\mu^6\cdot(x^6 - x_0^6)} \tag{31}$$

where μ^6 and x^6 are the 'lifted' points in $\mathbb{R}^6$ from μ and x respectively. In our case will μ^6 be of type b (see (35) and (36)). This leads to the following table of values.

	a_0	b_0	c_0
$\chi(a)$	1	−1	1
$\chi(b)$	−1	1	−1
$\chi(c)$	1	−1	1

(32)

From the *non-overlap condition* the restrictions on the absolute value for allowed μ^*'s in orthogonal space in all three-fold directions are

$$|\mu^*| < \tfrac{2}{3}\tau(\tau+2)③ \approx 0.398 \tag{33}$$

and in all five-fold directions

$$|\mu^*| < \frac{2\tau^4}{\tau+2}⑤ \approx 0.373. \tag{34}$$

These restrictions define the window, an icosahedron truncated by the planes of a dodecahedron (or the other way round) in μ^*-(orthogonal) space, i.e. a quasilattice, or Meyer set in dual (reciprocal) space. For the purpose of the selection of the points μ, it suffices to approximate the window P by a ball with radius $R = 0.373$. From the corresponding quasilattice we choose one μ for each of the 20 (10) three-fold directions

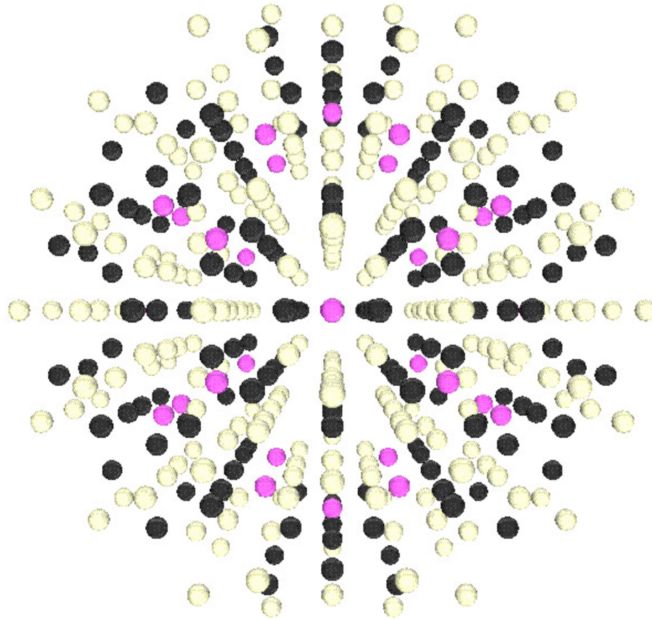

Figure 8. Icosahedrally symmetric quasilattice around a hole of type b_0. The corresponding starting local configuration $S(b_0)$ are 12 vectors in the five-fold directions of length $\tau⑤$, and 20 vectors $\tau③$.

(This figure can be viewed in colour in the electronic version of the article; see http://www.iop.org)

and one μ for each of the 12 (6) five-fold directions. For convenience we choose them both to be of type b. The chosen representative μ in $\mathbb{E}_\parallel$ along a three-fold direction is

$$\mu = \frac{\tau^5}{\sqrt{3}}\left(\tau, \frac{1}{\tau}, 0\right)③ \tag{35}$$

and in the five-fold direction

$$\mu = \frac{\tau^3}{\sqrt{\tau+2}}(1, \tau, 0)⑤ \tag{36}$$

see figure 7. The values of ε depend on which type of hole ($x = a$, b or c) is to be tested for acceptance. For the window condition (see the windows in figure 6) and the corresponding μ (see above) we determine the ε for the inequality of type (7)

$$\varepsilon_a = 2\sin\frac{\pi}{2(\tau+2)} \approx 0.841 \tag{37}$$

$$\varepsilon_b = 2\sin\frac{\pi}{2\tau^2(\tau+2)} \approx 0.330 \tag{38}$$

$$\varepsilon_c = 2\sin\frac{\pi}{2\tau(\tau+2)} \approx 0.530. \tag{39}$$

In figures 8–10 we present some of the examples of the $\mathcal{T}^{(2F)}$ quasilattices constructed by the method. Holes of type a are black, of type b are grey and of type c are white.

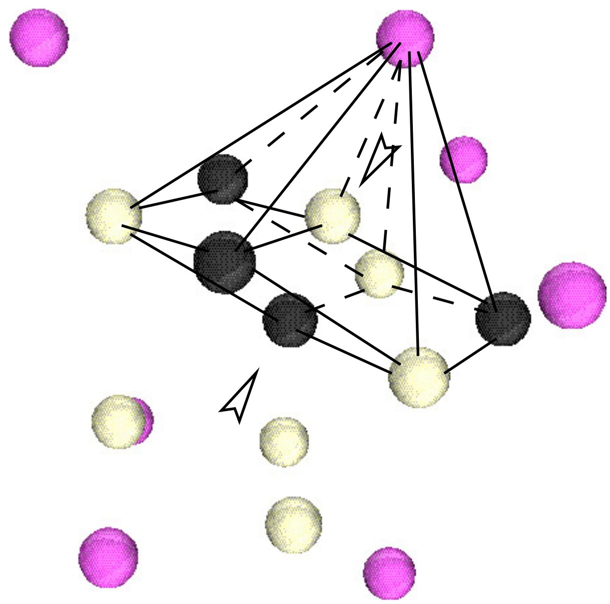

Figure 9. A quasilattice without icosahedral symmetry: the starting seed are two holes, one of type c_0 and another of type a_0.

(This figure can be viewed in colour in the electronic version of the article; see `http://www.iop.org`)

7. Conclusion and outlook

We have generated two quasilattices associated with the icosahedral F–phase by the coherent phase method in the physical space. The outward growth proceeds along the edge vectors of the ideal local configuration in the form of vector stars along axes of the icosahedral group. We determine points μ in reciprocal space which index the finite set of continuous characters that are used and give the controlling parameters ε. In the first case the growth propagates along two-fold axes and generates the vertex set of the tiling $\mathcal{T}^{*(2F)}$. In the second case it propagates along three-fold and five-fold axes and generates the vertex set (with three types of points) of the tiling $\mathcal{T}^{(2F)}$.

Note that not all the edge vectors of the local configurations used in the growth process are edges of the tiling, compare with figure 5. For the Penrose quasilattice, de Bruijn [10] has shown that the vertex set uniquely determines the Penrose tiling. This quasilattice is obtained by the projection of the holes in the lattice A_4, using as windows in $\mathbb{E}_\perp$ the orthogonal projected Delaunay cells. In [11] it will be shown that similarly the vertex set of the tiling $\mathcal{T}^{(2F)}$ determines the full tiling. The case of $\mathcal{T}^{*(2F)}$ is more complicated. There is the tiling of Mosseri and Sadoc [12] that can be locally derived from the tiling $\mathcal{T}^{*(2F)}$ [13]. From the vertex set of the tiling $\mathcal{T}^{*(2F)}$, one can reconstruct the Mosseri and Sadoc tiling, but not the tiling $\mathcal{T}^{*(2F)}$ itself.

The generation of the canonical quasicrystal tilings by the coherent phase method [2] should be seen as the first indepth three-dimensional examples of potentially physically interesting quasicrystals by this new constructive mechanism. What makes the method quite different from all other established methods (inflation, projection method etc) is that it is based on the physically observable $k_\parallel$-(μ-)space.

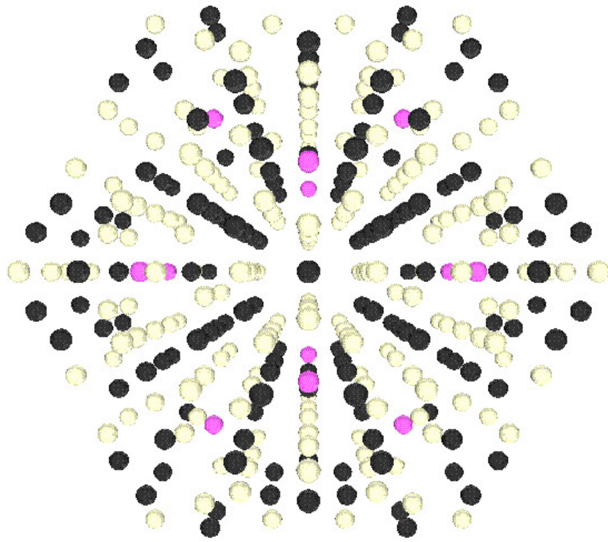

Figure 10. A quasilattice without icosahedral symmetry: the starting seed is a hole of type a_0. γ is small, the corresponding starting configuration $S(a_0)$ are 12 vectors in the five-fold directions of length ⑤, the same as in the icosahedrally symmetric case.

(This figure can be viewed in colour in the electronic version of the article; see `http://www.iop.org`)

Taking into account the size of the windows in $x_\perp$-space and the shortest vectors of ideal local configuration in $x_\parallel$-space through the non-overlap condition, the method explicitly yields a window in $k_\perp$-space. This, in turn, determines a quasilattice in $k_\parallel$-space (with a minimal spacing). Moreover, it is found (in calculations for delta-scatterers all of the same strength) that these $k_\parallel$-vectors correspond to strong Bragg peaks. The k-quasilattice so defined may then also be a candidate for the discussion of electronic properties of quasicrystals in terms of k-space, see [14] and references therein.

It would be interesting to see the method applied as a constructive formulation for some atomic models of quasicrystals, for example, the 'cluster models', see [15, 16] and references therein. The ideal local configuration could be determined from the suggested cluster structure in $x_\parallel$-space. The ε-values could be calculated from the experimentally obtained windows in $x_\perp$-space and from a finite set of continuous characters. The latter could be taken from a set of strong Bragg peaks. One could construct $k_\parallel$-space as a quasilattice, taking into account both the experimentally defined windows and the shortest distances of the atoms taken in various (icosahedral) directions. Finally, one could compare this obtained k-quasilattice with the experimentally observed positions of the strong diffraction peaks.

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